Assignment 3 for MATH4220

February 22, 2018

(No need to hand in.)

Exercise 3.1

1. Solve $u_t = ku_{xx}$; $u(x,0) = e^{-x}$; $u(0,t) = 0$ on the half-line $0 < x < \infty$.

2. Solve $u_t = ku_{xx}$; $u(x,0) = 0$; $u(0,t) = 1$ on the half-line $0 < x < \infty$.

3. Derive the solution formula for the half-line Neumann problem $w_t - kw_{xx} = 0$ for $0 < x < \infty, 0 < t < \infty$; $w_x(0,t) = 0$; $w(x,0) = \phi(x)$.

4. Consider the following problem with a Robin boundary condition:

$$\begin{aligned} \text{DE}: &\quad u_t = ku_{xx} && \text{on the half line } 0 < x < \infty, 0 < t < \infty \\ \text{IC}: &\quad u(x,0) = x && \text{for } t = 0 \text{ and } 0 < x < \infty \\ \text{BC}: &\quad u_x(0,t) - 2u(0,t) = 0 && \text{for } x = 0. \end{aligned} \qquad (*)$$

   The purpose of this exercise is to verify the solution for $(*)$. Let $f(x) = x$ for $x > 0$, let $f(x) = x + 1 - e^{2x}$ for $x < 0$, and let

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$

   (a) What PDE and initial condition does $v(x,t)$ satisfy for $-\infty < x < \infty$?

   (b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x,t)$ satisfy for $-\infty < x < \infty$?

   (c) Show that $f'(x) - 2f(x)$ is an odd function (for $x \neq 0$).

   (d) Use Exercise 2.4.11 to show that w is an odd function of x.

   (e) Deduce that $v(x,t)$ satisfies $(*)$ for $x > 0$. Assuming uniqueness, deduce that the solution of $(*)$ is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$

Exercise 3.2

1. Solve the Neumann problem for the wave equation on the half-line $0 < x < \infty$.

2. The longitudinal vibrations of a semi-infinite flexible rod satisfy the wave equation $u_{tt} = c^2 u_{xx}$ for $x > 0$. Assume that the end $x = 0$ is free ($u_x = 0$); it is initially at rest but has a constant initial velocity V for $a < x < 2a$ and has zero initial velocity elsewhere. Plot u versus x at the times $t = 0, a/c, 3a/2c, 2a/c$, and $3a/c$.

3. A wave $f(x + ct)$ travels along a semi-infinite string ($0 < x < \infty$) for $t < 0$. Find the vibrations $u(x,t)$ of the string for $t > 0$ if the end $x = 0$ is fixed.

5. Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty$, $u(0,t) = 0$, $u(x,0) \equiv 1$, $u_t(x,0) \equiv 0$ using the reflection method. This solution has a singularity; find its location.

6. Solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \infty, 0 \leq t < \infty, u(x,0) = 0, u_t(x,0) = V$,

$$u_t(0,t) + au_x(0,t) = 0,$$

   where V, a and c are positive constants and $a > c$.

Exercise 3.3

1. Solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition:

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \qquad (0 < x < \infty, 0 < t < \infty) \\ u(0,t) &= 0 \qquad u(x,0) = \phi(x) \end{aligned}$$

using the method of reflection.

2. Solve the completely inhomogeneous diffusion problem on the half-line

$$\begin{aligned} v_t - kv_{xx} &= f(x,t) \qquad (0 < x < \infty, 0 < t < \infty) \\ v(0,t) &= h(t) \qquad v(x,0) = \phi(x), \end{aligned}$$

by carrying out the subtracction method begun in the text.

3. Solve the inhomogeneous Neumann diffusion problem on the half-line

$$\begin{aligned} w_t - kw_{xx} &= 0 \qquad (0 < x < \infty, 0 < t < \infty) \\ w_x(0,t) &= h(t) \qquad w(x,0) = \phi(x), \end{aligned}$$

by the subtraction method indicated in the text.

Exercise 3.4

1. Solve $u_{tt} = c^2 u_{xx} + xt$, $u(x,0) = 0$, $u_t(x,0) = 0$.
2. Solve $u_{tt} = c^2 u_{xx} + e^{\alpha x}$, $u(x,0) = 0$, $u_t(x,0) = 0$.
3. Solve $u_{tt} = c^2 u_{xx} + \cos x$, $u(x,0) = \sin x$, $u_t(x,0) = 1 + x$.
4. Show that the solution of the inhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + f, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x),$$

is the sum of three terms, one each for f, ϕ, ψ.

5. Let $f(x,t)$ be any function and let $u(x,t) = (1/2c) \iint_\Delta f$, where Δ is the triangle of dependence. Verify directly by differentiation that

$$u_{tt} = c^2 u_{xx} + f \quad \text{and} \quad u(x,0) \equiv u_t(x,0) \equiv 0.$$

(*Hint:*Begin by writing the formula as the *iterated* integral

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,s)\; dyds$$

and differentiate with care using the rule in the Appendix. This exercise is not easy.)

8. Show that the source operator for the wave equation solves the problem

$$\mathscr{S}_{tt} - c^2 \mathscr{S}_{xx} = 0, \mathscr{S}(0) = 0, \mathscr{S}_t(0) = I,$$

where I is the identity operator.

9. Let $u(t) = \int_0^t \mathscr{S}(t-s) f(s)\; ds$. Using *only* Exercise 8, show that u solves the inhomogeneous wave equation with zero initial data.

12. Derive the solution of the fully inhomogeneous wave equation on the half-line

$$v_{tt} - c^2 v_{xx} = f(x,t) \quad \text{in } 0 < x < \infty$$

$$v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x)$$

$$v(0,t) = h(t),$$

by means of the method using Green's theorem.(*Hint:* Integrate over the domain of dependence.)

13. Solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \infty$, $u(0,t) = t^2$, $u(x,0) = x$, $u_t(x,0) = 0$.

14. Solve the homogeneous wave equation on the half-line $(0, \infty)$ with zero initial data and with the Neumann boundary condition $u_x(0,t) = k(t)$. Use any method you wish.

Exercise 3.5

1. Prove that if ϕ is any piecewise continuous function, then

$$\frac{1}{\sqrt{4\pi}} \int_0^{\pm\infty} e^{-p^2/4} \phi(x + \sqrt{kt}p)\, dp \to \pm\frac{1}{2}\phi(x\pm) \quad \text{as } t \searrow 0.$$

2. Use Exercise 1 to prove Theorem 2.